Localization of a polymer in a random environment - Bernoulli & Interlacements

Nicolas Bouchot

Sorbonne Université, LPSM

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k_B is the Boltzmann constant, T is the temperature of the system and Z is a normalizing constant (partition function).

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Polymer measure: Gibbs transformation of $\mathbf{P}$ given by

$$\mathbf{P}_n^\beta(S) = \frac{1}{Z_n^\beta} e^{-\beta H_n(S)} \mathbf{P}(S)$$

with $\beta = 1/k_B T$ and $Z_n^\beta = \mathbf{E}\left[e^{-\beta H_n(S)}\right]$ a normalizing constant.

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$$\mathbb{P}_n^{\beta}(S) = \frac{1}{\mathbb{Z}_n^{\beta}} \mathbb{E}\left[e^{-\beta H_n^\omega(S)}\right] \mathbf{P}(S)$$

where the environment plays a part in the equilibrium (compromise).

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$$\mathbb{P}_n^p(S) = \frac{1}{\mathbb{Z}_n^p} \mathbb{E}_p \left[\mathbb{1}_{\{S \cap \mathcal{O}_p = \varnothing\}} \right] \mathbf{P}(S) = \frac{1}{\mathbb{Z}_n^p} e^{|\mathcal{R}_n(S)| \log(1-p)} \mathbf{P}(S)$$

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$$\mathbb{P}^{h,\beta}_{n,\omega}(S) = \frac{1}{\mathbb{Z}^{h,\beta}_{n,\omega}} \prod_{z\in\mathcal{R}_n(S)} (1-p_z)\mathbf{P}(S) = \frac{1}{\mathbb{Z}^{h,\beta}_{n,\omega}} e^{\sum_{z\in\mathcal{R}_n(S)}(\beta\omega_z - h)}\mathbf{P}(S)$$

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This model is the random walk penalized by the sum of $h - \beta\omega_z$ sitting on the range, with quenched ω_z.

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We are interested in the edges localization, meaning asymptotics for

$$M_n^- = \min_{k\leq n} S_k \ , \ M_n^+ = \max_{k\leq n} S_k$$

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Define $T_n = M_n^+ - M_n^-$, $T_n^* = \left(\frac{n\pi^2}{h_n}\right)^{1/3}$ and $a_n = \frac{(T_n^*)^2}{\sqrt{3n\pi^2}} = \frac{1}{\sqrt{3}}\left(\frac{n\pi^2}{h_n^4}\right)^{1/6}$

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Theorem

- *Assume that* $h_n \geq n^{-1/2}(\log n)^{3/2}$ *and* $\lim_{n\to\infty} n^{-1/4} h_n = 0$, *then*

$$\left(\frac{T_n - T_n^*}{a_n}, \frac{M_n^+}{T_n^*}\right) \xrightarrow[n\to+\infty]{(d)} (\mathcal{T}, \mathcal{M}),$$

where $\mathcal{T} \sim \mathcal{N}(0,1)$ *and* $\mathcal{M} \sim \frac{\pi}{2}\sin(\pi y)\mathbb{1}_{[0,1]}(y)dy$ *are independent.*

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- *Assume that* $\lim_{n\to\infty} n^{-1/4} h_n = +\infty$ *and* $\lim_{n\to\infty} n^{-1} h_n = 0$.

$$\exists (\mathcal{A}_n)_{n\geq 1} \subseteq \{0,1\}, \quad \lim_{n\to\infty} \mathbf{P}_{n,h_n}\left(T_n - \lfloor T_n^* - 2 \rfloor \notin \mathcal{A}_n\right) = 0.$$

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Theorem (Berger, Huang, Torri, Wei (2022))

For all $h > 0$, define $c_h := (\pi^2 h^{-1})^{1/3}$. Then, for any $h, \beta > 0$, $\mathbb{P}$-almost surely we have the following convergence

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \log Z_{n,h}^{\omega,\beta} = -\frac{3}{2}(\pi h)^{2/3}, \qquad n^{-1/3}|\mathcal{R}_n| \xrightarrow[n\to\infty]{\mathbf{P}_{n,h}^{\omega,\beta}} c_h\,.$$

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The main contribution to $Z_{n,h}^{\omega,\beta}$ is given by trajectories with range $\sim c_h n^{1/3}$.

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$$\lim_{n\to\infty} \frac{1}{\beta n^{1/6}} \left(\log Z_{n,h}^{\omega,\beta} + \frac{3}{2} h c_h n^{1/3} \right) = \sup_{0 \le u \le c_h} \left\{ X_u^{(1)} + X_{c_h - u}^{(2)} \right\},$$

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with $X^{(1)}, X^{(2)}$ two independent Brownian motions.
Furthermore, $u_* := \arg\max_{u\in[0,c_h]} X_u^{(1)} + X_{c_h-u}^{(2)}$ is $\mathbb{P}$-a.s. unique and

$$n^{-1/3}(M_n^-, M_n^+) \xrightarrow[n\to\infty]{\mathbf{P}_{n,h}^{\omega,\beta}} (-u_*, c_h - u_*) \qquad \mathbb{P} - a.s.$$

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The location of the range is **P**-deterministic: it only depends on the realisation ω through $X_u = X_u^{(1)} + X_{c_h-u}^{(2)}$.

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Theorem

Suppose $\mathbb{E}\left[|\omega_0|^{3+\eta}\right] < \infty$ *for some positive* η*, we have the* $\mathbb{P}$*-a.s. convergence*

$$\lim_{n\to\infty} \frac{\sqrt{2}}{\beta n^{1/9}} \left(\log Z_{n,h}^{\omega,\beta} + \frac{3}{2} h c_h n^{1/3} - \beta n^{1/6} X_{u_*}\right) = \sup_{u,v} \left\{\mathcal{Y}_{u,v} - c_{h,\beta}(u+v)^2\right\}$$

where $\mathcal{Y}_{u,v} := \mathbf{Y}_u - \mathbf{Y}_{-v} - \chi(\mathbf{B}_u + \mathbf{B}_v)$ *with* $\mathbf{B}$ *a two-sided* BES_3*,* $\mathbf{Y}$ *a two-sided standard Brownian motion.*

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$$\lim_{n\to\infty} \frac{\sqrt{2}}{\beta n^{1/9}} \left(\log Z_{n,h}^{\omega,\beta} + \frac{3}{2} h c_h n^{1/3} - \beta n^{1/6} X_{u_*} \right) = \sup_{u,v} \left\{ \mathcal{Y}_{u,v} - c_{h,\beta} (u+v)^2 \right\}$$

where $\mathcal{Y}_{u,v} := \mathbf{Y}_u - \mathbf{Y}_{-v} - \chi(\mathbf{B}_u + \mathbf{B}_v)$ *with* $\mathbf{B}$ *a two-sided* BES_3*,* $\mathbf{Y}$ *a two-sided standard Brownian motion.*

Moreover, $(\mathcal{U}, \mathcal{V}) := \arg\max_{u,v} \{\mathcal{Y}_{u,v} - c_{h,\beta}(u+v)^2\}$ *is* $\mathbb{P}$*-a.s. well-defined, unique, and we have*

$$\frac{1}{n^{2/9}} (U_n, V_n) \xrightarrow[n\to\infty]{\mathbf{P}_{n,h}^{\omega,\beta}} (\mathcal{U}, \mathcal{V}) \quad \mathbb{P} - a.s.$$

Fluctuations around u_*

What is the scale/law of $U_n = M_n^- + u_* n^{1/3}$ or $V_n = (c_h - u_*) n^{1/3} - M_n^+$?

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Again, this is **P**-deterministic.

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Theorem (Ding, Xu, 2019)

Write τ for the hitting time of $\mathcal{O}_p$ then there exist $\mathbb{P}$-measurable regions $\mathcal{U}_n$, distant from at least $n(\log n)^{-\gamma}$ and with poly-logarithmic volume, and a random time $T_n = o(n)$ such that $\mathbf{P}\left(S_{[T,n]\subseteq \mathcal{U}_n} \,\middle|\, \tau > n\right) \to 1$.

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$\mathscr{I}^u$ is a random subset of $\mathbb{Z}^d$ caracterized by

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with $\mathrm{cap}\,[K] = \sum_{x \in K} \mathbf{P}_x \left(S_{[1,+\infty]} \cap K = \varnothing \right)$ the capacity of $K \subset \mathbb{Z}^d$.

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Conjecture: There exist almost-empty regions $\mathscr{R}$ at distance $n(\log n)^{-\mu}$ with size $(\log n)^{\lambda}$ such that $S_{[o(n),n]} \subseteq \mathscr{R}$.

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Mostly done with the same ideas as Ding & Xu.

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The annealed partition function of $H_n^u = \infty \mathbb{1}_{\{\mathcal{R}_n \cap \mathscr{I}^u\}}$ is given by

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We conjecture a phase diagram for $\mathrm{cap}\,[R_n]$ when taking $u = u(n)$ depending on the asymptotics of $u(n)$.

Thank you for your attention!