Confined random walk and tilted interlacements

Nicolas Bouchot
joint work with Quentin Berger & Julien Poisat

LPSM (Sorbonne Université) - CEREMADE (Université Paris Dauphine PSL)

Geometry of random walks
Stochastic Processes & Applications - Wrocław 2025

Simple random walk trapped in a large domain

Simple random walk trapped in a large domain

Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.

Simple random walk trapped in a large domain

Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.
Let $\mathbf{P}$ be the law of SRW on $\mathbb{Z}^d$ and $\mathcal{R}_t := \{S_0, \dots, S_t\}$.

Simple random walk trapped in a large domain

Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.
Let $\mathbf{P}$ be the law of SRW on $\mathbb{Z}^d$ and $\mathcal{R}_t := \{S_0, \dots, S_t\}$.

Question: what is the local geometry of $\mathcal{R}_{t_N}$ under $\mathbf{P}(\cdot \,|\, \mathcal{R}_{t_N} \subseteq D_N)$?

Simple random walk trapped in a large domain

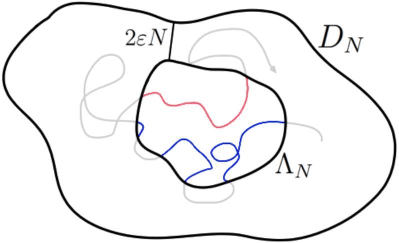
Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.
Let $\mathbf{P}$ be the law of SRW on $\mathbb{Z}^d$ and $\mathcal{R}_t := \{S_0, \dots, S_t\}$.

Question: what is the local geometry of $\mathcal{R}_{t_N}$ under $\mathbf{P}(\cdot \,|\, \mathcal{R}_{t_N} \subseteq D_N)$?
We look at $\mathcal{R}_{t_N} \cap \Lambda_N$ with $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d \subset D_N$.

Simple random walk trapped in a large domain

Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.
Let $\mathbf{P}$ be the law of SRW on $\mathbb{Z}^d$ and $\mathcal{R}_t := \{S_0, \ldots, S_t\}$.

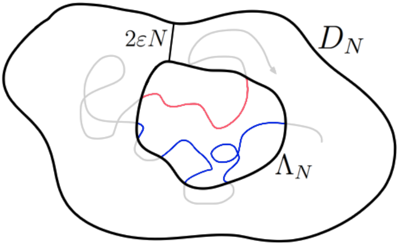
Question: what is the local geometry of $\mathcal{R}_{t_N}$ under $\mathbf{P}(\cdot \,|\, \mathcal{R}_{t_N} \subseteq D_N)$?
We look at $\mathcal{R}_{t_N} \cap \Lambda_N$ with $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d \subset D_N$.



Simple random walk trapped in a large domain

Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.
Let $\mathbf{P}$ be the law of SRW on $\mathbb{Z}^d$ and $\mathcal{R}_t := \{S_0, \dots, S_t\}$.

Question: what is the local geometry of $\mathcal{R}_{t_N}$ under $\mathbf{P}(\cdot \,|\, \mathcal{R}_{t_N} \subseteq D_N)$?
We look at $\mathcal{R}_{t_N} \cap \Lambda_N$ with $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d \subset D_N$.

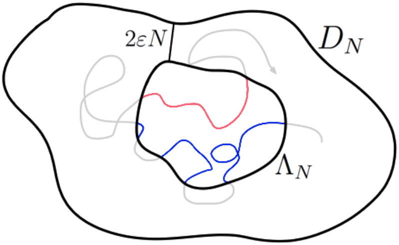


Two observations:

Simple random walk trapped in a large domain

Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.
Let $\mathbf{P}$ be the law of SRW on $\mathbb{Z}^d$ and $\mathcal{R}_t := \{S_0, \ldots, S_t\}$.

Question: what is the local geometry of $\mathcal{R}_{t_N}$ under $\mathbf{P}(\cdot \,|\, \mathcal{R}_{t_N} \subseteq D_N)$?
We look at $\mathcal{R}_{t_N} \cap \Lambda_N$ with $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d \subset D_N$.



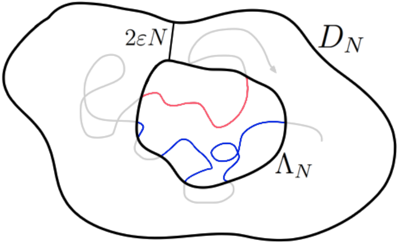
Two observations:

→ We can group excursions into mostly independent packs.

Simple random walk trapped in a large domain

Consider a large domain $D_N := (N \cdot D) \cap \mathbb{Z}^d$ where D is a "reasonable" bounded open set of $\mathbb{R}^d, d \geq 3$.
Let $\mathbf{P}$ be the law of SRW on $\mathbb{Z}^d$ and $\mathcal{R}_t := \{S_0, \ldots, S_t\}$.

Question: what is the local geometry of $\mathcal{R}_{t_N}$ under $\mathbf{P}(\cdot \,|\, \mathcal{R}_{t_N} \subseteq D_N)$?
We look at $\mathcal{R}_{t_N} \cap \Lambda_N$ with $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d \subset D_N$.



Two observations:

→ We can group excursions into mostly independent packs.

→ Conditioning creates an effective (time-dependent) drift.

The confined walk

The confined walk

By standard Markov theory, we can define the confined walk: a Markov chain with transitions $p_N(x,y) := \lim_{T \to +\infty} \mathbf{P}_x(S_1 = y \,|\, \mathcal{R}_T \subseteq D_N)$.

The confined walk

By standard Markov theory, we can define the confined walk: a Markov chain with transitions $p_N(x,y) := \lim_{T \to +\infty} \mathbf{P}_x(S_1 = y \,|\, \mathcal{R}_T \subseteq D_N)$.

Write $\mathrm{P}_N = \mathrm{P}_{D_N}$ the transition matrix of SRW killed on ∂D_N:

$$\forall x, y \in D_N, \quad \mathrm{P}_N(x,y) = \tfrac{1}{2d} \mathbb{1}_{\{x \sim y\}}.$$

The confined walk

By standard Markov theory, we can define the confined walk: a Markov chain with transitions $p_N(x,y) := \lim_{T\to+\infty} \mathbf{P}_x(S_1 = y \,|\, \mathcal{R}_T \subseteq D_N)$.

Write $\mathrm{P}_N = \mathrm{P}_{D_N}$ the transition matrix of SRW killed on ∂D_N:
$\forall x, y \in D_N, \quad \mathrm{P}_N(x,y) = \frac{1}{2d}\mathbb{1}_{\{x\sim y\}}$.

Perron-Frobenius: λ_N largest eigenvalue of P_N, with eigenvector ϕ_N.

$$\mathrm{P}_N\phi_N = \lambda_N\phi_N \quad , \quad \frac{1}{N^d}\sum_{z\in D_N}\phi_N^2(z) = 1\,.$$

The confined walk

By standard Markov theory, we can define the confined walk: a Markov chain with transitions $p_N(x,y) := \lim_{T\to+\infty} \mathbf{P}_x(S_1 = y \,|\, \mathcal{R}_T \subseteq D_N)$.

Write $\mathrm{P}_N = \mathrm{P}_{D_N}$ the transition matrix of SRW killed on ∂D_N:
$\forall x, y \in D_N, \quad \mathrm{P}_N(x,y) = \frac{1}{2d}\mathbb{1}_{\{x\sim y\}}$.

Perron-Frobenius: λ_N largest eigenvalue of P_N, with eigenvector ϕ_N.

$$\mathrm{P}_N\phi_N = \lambda_N\phi_N \quad , \quad \frac{1}{N^d}\sum_{z\in D_N}\phi_N^2(z) = 1\,.$$

We then have

$$p_N(x,y) = \frac{\lambda_N^{-1}}{2d}\frac{\phi_N(y)}{\phi_N(x)}\mathbb{1}_{\{x\sim y\}} = \frac{1}{2d}\frac{\phi_N(y)}{\sum_{z\sim x}\phi_N(z)}\mathbb{1}_{\{x\sim y\}}\,.$$

The confined walk

By standard Markov theory, we can define the confined walk: a Markov chain with transitions $p_N(x,y) := \lim_{T\to+\infty} \mathbf{P}_x(S_1 = y \,|\, \mathcal{R}_T \subseteq D_N)$.

Write $\mathrm{P}_N = \mathrm{P}_{D_N}$ the transition matrix of SRW killed on ∂D_N:
$\forall x,y \in D_N, \quad \mathrm{P}_N(x,y) = \frac{1}{2d}\mathbb{1}_{\{x\sim y\}}$.

Perron-Frobenius: λ_N largest eigenvalue of P_N, with eigenvector ϕ_N.

$$\mathrm{P}_N\phi_N = \lambda_N\phi_N \quad , \quad \frac{1}{N^d}\sum_{z\in D_N}\phi_N^2(z) = 1\,.$$

We then have

$$p_N(x,y) = \frac{\lambda_N^{-1}}{2d}\frac{\phi_N(y)}{\phi_N(x)}\mathbb{1}_{\{x\sim y\}} = \frac{1}{2d}\frac{\phi_N(y)}{\sum_{z\sim x}\phi_N(z)}\mathbb{1}_{\{x\sim y\}}\,.$$

The confined walk is the random walk on the conductances
$c_N(x,y) = \phi_N(x)\phi_N(y)$.

The confined walk

By standard Markov theory, we can define the confined walk: a Markov chain with transitions $p_N(x,y) := \lim_{T\to+\infty} \mathbf{P}_x(S_1 = y \,|\, \mathcal{R}_T \subseteq D_N)$.

Write $\mathrm{P}_N = \mathrm{P}_{D_N}$ the transition matrix of SRW killed on ∂D_N:
$\forall x, y \in D_N, \quad \mathrm{P}_N(x,y) = \frac{1}{2d}\mathbb{1}_{\{x\sim y\}}$.

Perron-Frobenius: λ_N largest eigenvalue of P_N, with eigenvector ϕ_N.

$$\mathrm{P}_N\phi_N = \lambda_N\phi_N \quad , \quad \frac{1}{N^d}\sum_{z\in D_N}\phi_N^2(z) = 1\,.$$

We then have

$$p_N(x,y) = \frac{\lambda_N^{-1}}{2d}\frac{\phi_N(y)}{\phi_N(x)}\mathbb{1}_{\{x\sim y\}} = \frac{1}{2d}\frac{\phi_N(y)}{\sum_{z\sim x}\phi_N(z)}\mathbb{1}_{\{x\sim y\}}\,.$$

The confined walk is the random walk on the conductances
$c_N(x,y) = \phi_N(x)\phi_N(y)$.
We write $\mathcal{R}_{\phi_N}(t_N)$ its range up to time t_N.

Random interlacements

Random interlacements

Let W be the set of bi-infinite transient nearest-neighbour paths in $\mathbb{Z}^d$:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{Z}^d \;:\; \forall i \in \mathbb{Z},\, w_{i+1} \sim w_i \text{ and } \lim_{|i| \to +\infty} |w_i| = +\infty \right\}.$$

and let $\mathcal{W}$ stand for W quotiented by time shift.

Random interlacements

Let W be the set of bi-infinite transient nearest-neighbour paths in $\mathbb{Z}^d$:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{Z}^d \,:\, \forall i \in \mathbb{Z},\, w_{i+1} \sim w_i \text{ and } \lim_{|i| \to +\infty} |w_i| = +\infty \right\}.$$

and let $\mathcal{W}$ stand for W quotiented by time shift.
One can define a measure ν which is "uniform" on $\mathcal{W}$.

Random interlacements

Let W be the set of bi-infinite transient nearest-neighbour paths in $\mathbb{Z}^d$:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{Z}^d \ : \ \forall i \in \mathbb{Z},\ w_{i+1} \sim w_i \text{ and } \lim_{|i| \to +\infty} |w_i| = +\infty \right\}.$$

and let $\mathcal{W}$ stand for W quotiented by time shift.
One can define a measure ν which is “uniform” on $\mathcal{W}$.

Definition

The random interlacements is the trace of a Poisson point process χ on $\mathcal{W} \times \mathbb{R}_+$ with intensity measure $\nu \otimes du$:

Random interlacements

Let W be the set of bi-infinite transient nearest-neighbour paths in $\mathbb{Z}^d$:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{Z}^d \ : \ \forall i \in \mathbb{Z},\ w_{i+1} \sim w_i \text{ and } \lim_{|i| \to +\infty} |w_i| = +\infty \right\}.$$

and let $\mathcal{W}$ stand for W quotiented by time shift.
One can define a measure ν which is "uniform" on $\mathcal{W}$.

Definition

The random interlacements is the trace of a Poisson point process χ on $\mathcal{W} \times \mathbb{R}_+$ with intensity measure $\nu \otimes du$:

$$\chi = \sum_j \delta_{(w^j, u_j)} \quad , \quad \mathscr{I}(u) = \left\{ w^j(\mathbb{Z}) \ : \ (w^j, u_j) \in \chi \text{ and } u_j \leq u \right\}.$$

Random interlacements

Let W be the set of bi-infinite transient nearest-neighbour paths in $\mathbb{Z}^d$:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{Z}^d \,:\, \forall i \in \mathbb{Z},\, w_{i+1} \sim w_i \text{ and } \lim_{|i| \to +\infty} |w_i| = +\infty \right\}.$$

and let $\mathcal{W}$ stand for W quotiented by time shift.
One can define a measure ν which is "uniform" on $\mathcal{W}$.

Definition

The random interlacements is the trace of a Poisson point process χ on $\mathcal{W} \times \mathbb{R}_+$ with intensity measure $\nu \otimes du$:

$$\chi = \sum_j \delta_{(w^j, u_j)} \quad , \quad \mathscr{I}(u) = \left\{ w^j(\mathbb{Z}) \,:\, (w^j, u_j) \in \chi \text{ and } u_j \le u \right\}.$$

For $h : \mathbb{Z}^d \longrightarrow (0, +\infty)$, we can define a measure ν^h on weighted paths with edge weights $c^h(x,y) = h(x)h(y)$, and the corresponding $\mathscr{I}_h(u)$.

Random interlacements

Let W be the set of bi-infinite transient nearest-neighbour paths in $\mathbb{Z}^d$:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{Z}^d \ : \forall i \in \mathbb{Z},\ w_{i+1} \sim w_i \text{ and } \lim_{|i| \to +\infty} |w_i| = +\infty \right\}.$$

and let $\mathcal{W}$ stand for W quotiented by time shift.
One can define a measure ν which is “uniform” on $\mathcal{W}$.

Definition

The random interlacements is the trace of a Poisson point process χ on $\mathcal{W} \times \mathbb{R}_+$ with intensity measure $\nu \otimes du$:

$$\chi = \sum_j \delta_{(w^j, u_j)} \quad , \quad \mathscr{I}(u) = \left\{ w^j(\mathbb{Z}) \ : \ (w^j, u_j) \in \chi \text{ and } u_j \leq u \right\}.$$

For $h : \mathbb{Z}^d \longrightarrow (0, +\infty)$, we can define a measure ν^h on weighted paths with edge weights $c^h(x,y) = h(x)h(y)$, and the corresponding $\mathscr{I}_h(u)$.

We want $h = \phi_N$ near Λ_N, but we need to “extend” ϕ_N to $\mathbb{Z}^d$.

Statement of the coupling

Statement of the coupling

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \left\{ x \in D_N \,:\, d(x, \Lambda_N) \leq \varepsilon N \right\}$, and 1 elsewhere.

Statement of the coupling

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \left\{x \in D_N \,:\, d(x, \Lambda_N) \leq \varepsilon N\right\}$, and 1 elsewhere.

Theorem (B. 2025+ – arXiv:2405.14329)

Statement of the coupling

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \left\{x \in D_N \,:\, d(x, \Lambda_N) \leq \varepsilon N\right\}$, and 1 elsewhere.

Theorem (B. 2025+ – arXiv:2405.14329)

Let $\delta \in (0,1)$ and assume $t_N \gg N^{2+\delta}$.

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \left\{x \in D_N \,:\, d(x, \Lambda_N) \leq \varepsilon N\right\}$, and 1 elsewhere.

Theorem (B. 2025+ – arXiv:2405.14329)

Let $\delta \in (0,1)$ and assume $t_N \gg N^{2+\delta}$. We set

$$u_N := t_N/\lambda_N N^d \quad , \quad \varepsilon_N = N^{-\delta/4}.$$

Statement of the coupling

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \{x \in D_N \,:\, d(x, \Lambda_N) \leq \varepsilon N\}$, and 1 elsewhere.

Theorem (B. 2025+ – arXiv:2405.14329)

Let $\delta \in (0,1)$ and assume $t_N \gg N^{2+\delta}$. We set

$$u_N := t_N / \lambda_N N^d \quad , \quad \varepsilon_N = N^{-\delta/4} \, .$$

Then, there is a coupling $\mathbb{Q}_N$ of $\mathcal{R}_{\phi_N}(t_N)$ and $\mathscr{I}_{\Psi_N}((1 \pm \varepsilon_N) u_N)$ such that, for all N large enough we have

Statement of the coupling

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \{x \in D_N \, : \, d(x, \Lambda_N) \leq \varepsilon N\}$, and 1 elsewhere.

Theorem (B. 2025+ – arXiv:2405.14329)

Let $\delta \in (0,1)$ and assume $t_N \gg N^{2+\delta}$. We set

$$u_N := t_N / \lambda_N N^d \quad , \quad \varepsilon_N = N^{-\delta/4}\,.$$

Then, there is a coupling $\mathbb{Q}_N$ of $\mathcal{R}_{\phi_N}(t_N)$ and $\mathscr{I}_{\Psi_N}((1 \pm \varepsilon_N)u_N)$ such that, for all N large enough we have

$$\mathscr{I}_{\Psi_N}((1-\varepsilon_N)u_N) \cap \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t_N) \cap \Lambda_N \subseteq \mathscr{I}_{\Psi_N}((1+\varepsilon_N)u_N) \cap \Lambda_N\,,$$

Statement of the coupling

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \left\{x \in D_N \,:\, d(x, \Lambda_N) \le \varepsilon N\right\}$, and 1 elsewhere.

Theorem (B. 2025+ – arXiv:2405.14329)

Let $\delta \in (0,1)$ and assume $t_N \gg N^{2+\delta}$. We set

$$u_N := t_N / \lambda_N N^d \quad , \quad \varepsilon_N = N^{-\delta/4} .$$

Then, there is a coupling $\mathbb{Q}_N$ of $\mathcal{R}_{\phi_N}(t_N)$ and $\mathscr{I}_{\Psi_N}((1 \pm \varepsilon_N) u_N)$ such that, for all N large enough we have

$$\mathscr{I}_{\Psi_N}((1-\varepsilon_N)u_N) \cap \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t_N) \cap \Lambda_N \subseteq \mathscr{I}_{\Psi_N}((1+\varepsilon_N)u_N) \cap \Lambda_N ,$$

with $\mathbb{Q}_N$-probability at least $1 - c_1 e^{-c_2 N^\eta}$.

Statement of the coupling

Define $\Psi_N = \phi_N$ on $\Lambda_N^\varepsilon = \{x \in D_N \,:\, d(x, \Lambda_N) \le \varepsilon N\}$, and 1 elsewhere.

Theorem (B. 2025+ – arXiv:2405.14329)

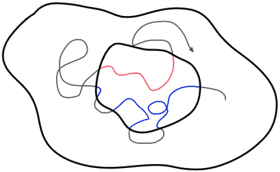
Let $\delta \in (0,1)$ and assume $t_N \gg N^{2+\delta}$. We set

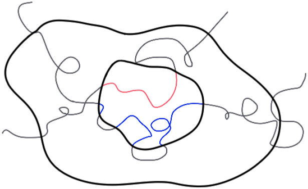
$$u_N := t_N / \lambda_N N^d \quad , \quad \varepsilon_N = N^{-\delta/4} .$$

Then, there is a coupling $\mathbb{Q}_N$ of $\mathcal{R}_{\phi_N}(t_N)$ and $\mathscr{I}_{\Psi_N}((1 \pm \varepsilon_N) u_N)$ such that, for all N large enough we have

$$\mathscr{I}_{\Psi_N}((1-\varepsilon_N)u_N) \cap \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t_N) \cap \Lambda_N \subseteq \mathscr{I}_{\Psi_N}((1+\varepsilon_N)u_N) \cap \Lambda_N ,$$

with $\mathbb{Q}_N$-probability at least $1 - c_1 e^{-c_2 N^\eta}$.





Excursion decomposition of random interlacements

Excursion decomposition of random interlacements

Consider $K \subset \mathbb{Z}^d$ finite, and define

$$\mathrm{cap}^h(K) = \sum_{x \in K} \mathbf{P}_x^h(S_1, S_2, \cdots \notin K)\,.$$

with $\mathbf{P}^h$ the law of the random walk on conductances $c^h(x,y) = h(x)h(y)$.

Consider $K \subset \mathbb{Z}^d$ finite, and define

$$\mathrm{cap}^h(K) = \sum_{x \in K} \mathbf{P}_x^h(S_1, S_2, \cdots \notin K)\,.$$

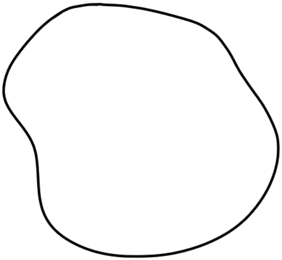
with $\mathbf{P}^h$ the law of the random walk on conductances $c^h(x,y) = h(x)h(y)$. $\mathscr{I}_h(u) \cap K$ can then be constructed in the following way:

Excursion decomposition of random interlacements

Consider $K \subset \mathbb{Z}^d$ finite, and define

$$\mathrm{cap}^h(K) = \sum_{x \in K} \mathbf{P}_x^h(S_1, S_2, \cdots \notin K)\,.$$

with $\mathbf{P}^h$ the law of the random walk on conductances $c^h(x,y) = h(x)h(y)$.
$\mathscr{I}_h(u) \cap K$ can then be constructed in the following way:



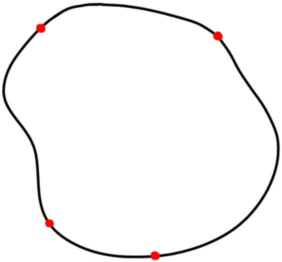
Draw $N_K^u \sim \mathcal{P}\big(u\mathrm{cap}(K)\big)$

Excursion decomposition of random interlacements

Consider $K \subset \mathbb{Z}^d$ finite, and define

$$\mathrm{cap}^h(K) = \sum_{x \in K} \mathbf{P}_x^h(S_1, S_2, \cdots \notin K)\,.$$

with $\mathbf{P}^h$ the law of the random walk on conductances $c^h(x,y) = h(x)h(y)$. $\mathscr{I}_h(u) \cap K$ can then be constructed in the following way:



Draw $N_K^u \sim \mathcal{P}\big(u\mathrm{cap}(K)\big)$

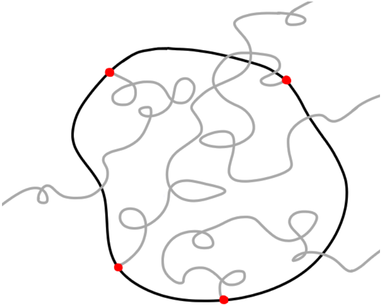
Sample $(X_0^j)_{j \leq N_K^u}$ i.i.d. according to $e_K(x) = \mathbf{P}_x^h(S_1, S_2, \cdots \notin K)$

Excursion decomposition of random interlacements

Consider $K \subset \mathbb{Z}^d$ finite, and define

$$\mathrm{cap}^h(K) = \sum_{x \in K} \mathbf{P}^h_x(S_1, S_2, \cdots \notin K)\,.$$

with $\mathbf{P}^h$ the law of the random walk on conductances $c^h(x,y) = h(x)h(y)$. $\mathscr{I}_h(u) \cap K$ can then be constructed in the following way:



Draw $N^u_K \sim \mathcal{P}\big(u\mathrm{cap}(K)\big)$

Sample $(X^j_0)_{j \le N^u_K}$ i.i.d. according to $e_K(x) = \mathbf{P}^h_x(S_1, S_2, \cdots \notin K)$

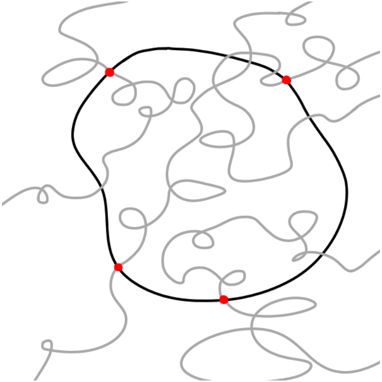
For each $j \in \{1, \dots, N^u_K\}$, start a RW from X^j_0 according to $\mathbf{P}^h_{X^j_0}$

Excursion decomposition of random interlacements

Consider $K \subset \mathbb{Z}^d$ finite, and define

$$\mathrm{cap}^h(K) = \sum_{x \in K} \mathbf{P}_x^h(S_1, S_2, \cdots \notin K)\,.$$

with $\mathbf{P}^h$ the law of the random walk on conductances $c^h(x,y) = h(x)h(y)$. $\mathscr{I}_h(u) \cap K$ can then be constructed in the following way:



Draw $N_K^u \sim \mathcal{P}\big(u\mathrm{cap}(K)\big)$

Sample $(X_0^j)_{j \le N_K^u}$ i.i.d. according to $e_K(x) = \mathbf{P}_x^h(S_1, S_2, \cdots \notin K)$

For each $j \in \{1, \ldots, N_K^u\}$, start a RW from X_0^j according to $\mathbf{P}^h_{X_0^j}$

For each $j \in \{1, \ldots, N_K^u\}$ start a RW according to $\mathbf{P}^h_{X_0^j}\big(\,\cdot \mid S_1, \cdots \notin K\big)$.

Coupling using soft local times

Coupling using soft local times

Introduced by Popov & Teixeira (2015) to decorrelate random interlacements.

Coupling using soft local times

Introduced by Popov & Teixeira (2015) to decorrelate random interlacements.

Main idea: construct the range of any Markov chain ζ on a finite state space Σ from a <u>single</u> Poisson point process η on $\Sigma \times \mathbb{R}_+$.

Coupling using soft local times

Introduced by Popov & Teixeira (2015) to decorrelate random interlacements.

Main idea: construct the range of any Markov chain ζ on a finite state space Σ from a single Poisson point process η on $\Sigma \times \mathbb{R}_+$.

Theorem (Černý, Teixeira ; 2016)

Consider ζ a Markov chain on Σ with invariant probability π, and (U_i) i.i.d. with law π. Then, there is a coupling such that

$$\mathbb{P}\Big(\{U_i\}_{i=1}^{(1-\varepsilon)n} \subseteq \{\zeta_i\}_{i=1}^{n} \subseteq \{U_i\}_{i=1}^{(1+\varepsilon)n} \Big) \geq 1 - \delta_\varepsilon^\zeta(n)\,.$$

Coupling using soft local times

Introduced by Popov & Teixeira (2015) to decorrelate random interlacements.

Main idea: construct the range of any Markov chain ζ on a finite state space Σ from a single Poisson point process η on $\Sigma \times \mathbb{R}_+$.

Theorem (Černý, Teixeira ; 2016)

Consider ζ a Markov chain on Σ with invariant probability π, and (U_i) i.i.d. with law π. Then, there is a coupling such that

$$\mathbb{P}\Big(\{U_i\}_{i=1}^{(1-\varepsilon)n} \subseteq \{\zeta_i\}_{i=1}^{n} \subseteq \{U_i\}_{i=1}^{(1+\varepsilon)n} \Big) \geq 1 - \delta_\varepsilon^\zeta(n)\,.$$

Markov chains here: entrance and exit points of paths inside Λ_N.

Coupling using soft local times

Introduced by Popov & Teixeira (2015) to decorrelate random interlacements.

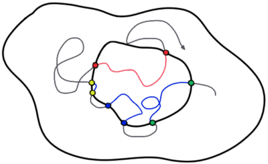
Main idea: construct the range of any Markov chain ζ on a finite state space Σ from a single Poisson point process η on $\Sigma \times \mathbb{R}_+$.

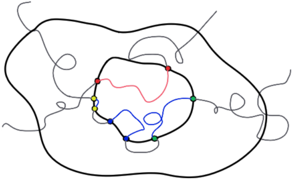
Theorem (Černý, Teixeira ; 2016)

Consider ζ a Markov chain on Σ with invariant probability π, and (U_i) i.i.d. with law π. Then, there is a coupling such that

$$\mathbb{P}\Big(\{U_i\}_{i=1}^{(1-\varepsilon)n} \subseteq \{\zeta_i\}_{i=1}^{n} \subseteq \{U_i\}_{i=1}^{(1+\varepsilon)n} \Big) \geq 1 - \delta_\varepsilon^\zeta(n)\,.$$

Markov chains here: entrance and exit points of paths inside Λ_N.





Coupling using soft local times

Introduced by Popov & Teixeira (2015) to decorrelate random interlacements.

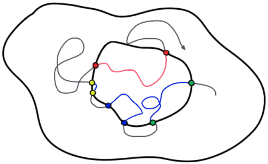
Main idea: construct the range of any Markov chain ζ on a finite state space Σ from a single Poisson point process η on $\Sigma \times \mathbb{R}_+$.

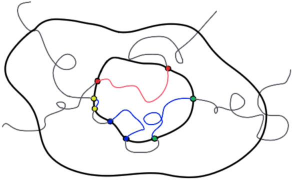
Theorem (Černý, Teixeira ; 2016)

Consider ζ a Markov chain on Σ with invariant probability π, and (U_i) i.i.d. with law π. Then, there is a coupling such that

$$\mathbb{P}\Big(\{U_i\}_{i=1}^{(1-\varepsilon)n} \subseteq \{\zeta_i\}_{i=1}^{n} \subseteq \{U_i\}_{i=1}^{(1+\varepsilon)n} \Big) \geq 1 - \delta_\varepsilon^\zeta(n)\,.$$

Markov chains here: entrance and exit points of paths inside Λ_N.





A sketch of the proof

A sketch of the proof

First task: determine the invariant measures and show that $\pi^Y \approx \pi^Z$.

A sketch of the proof

First task: determine the invariant measures and show that $\pi^Y \approx \pi^Z$.
$\rightarrow$ use the ergodic theorem for the confined walk, and the independence structure of interlacements.

A sketch of the proof

First task: determine the invariant measures and show that $\pi^Y \approx \pi^Z$.
$\to$ use the ergodic theorem for the confined walk, and the independence structure of interlacements.

The error of the coupling is then at most

$$\sum_{\circ=Y,Z} \sum_{z\in\Sigma} \left(e^{-cn\varepsilon_N^2} + e^{-cn\varepsilon_N \frac{\pi^\circ(z)}{\nu^\circ(z)}} + \exp\left(-\frac{cn}{T^\circ_{\mathrm{mix}}} \frac{\pi^\circ(z)^2 \varepsilon_N^2 / k(\varepsilon_N)}{\mathrm{Var}_{\pi^\circ}\left[p^\circ(\,\cdot\,,z)\right]} \right) \right)$$

A sketch of the proof

First task: determine the invariant measures and show that $\pi^Y \approx \pi^Z$.
$\rightarrow$ use the ergodic theorem for the confined walk, and the independence structure of interlacements.

The error of the coupling is then at most

$$\sum_{\circ=Y,Z}\sum_{z\in\Sigma}\left(e^{-cn\varepsilon_N^2}+e^{-cn\varepsilon_N\frac{\pi^\circ(z)}{\nu^\circ(z)}}+\exp\left(-\frac{cn}{T^\circ_{\text{mix}}}\frac{\pi^\circ(z)^2\varepsilon_N^2/k(\varepsilon_N)}{\mathrm{Var}_{\pi^\circ}\left[p^\circ(\,\cdot\,,z)\right]}\right)\right)$$

We need to get estimates on mixing times and variances.

A sketch of the proof

First task: determine the invariant measures and show that $\pi^Y \approx \pi^Z$.
$\rightarrow$ use the ergodic theorem for the confined walk, and the independence structure of interlacements.

The error of the coupling is then at most

$$\sum_{\circ=Y,Z} \sum_{z\in\Sigma} \left(e^{-cn\varepsilon_N^2} + e^{-cn\varepsilon_N \frac{\pi^\circ(z)}{\nu^\circ(z)}} + \exp\left(-\frac{cn}{T^\circ_{\mathrm{mix}}} \frac{\pi^\circ(z)^2 \varepsilon_N^2 / k(\varepsilon_N)}{\mathrm{Var}_{\pi^\circ}\left[p^\circ(\,\cdot\,, z)\right]} \right) \right)$$

We need to get estimates on mixing times and variances.
$\rightarrow$ use coalescent coupling and confined/tilted walk estimates.

A sketch of the proof

First task: determine the invariant measures and show that $\pi^Y \approx \pi^Z$.
$\to$ use the ergodic theorem for the confined walk, and the independence structure of interlacements.

The error of the coupling is then at most

$$\sum_{\circ=Y,Z}\sum_{z\in\Sigma}\left(e^{-cn\varepsilon_N^2}+e^{-cn\varepsilon_N\frac{\pi^\circ(z)}{\nu^\circ(z)}}+\exp\left(-\frac{cn}{T^\circ_{\mathrm{mix}}}\frac{\pi^\circ(z)^2\varepsilon_N^2/k(\varepsilon_N)}{\mathrm{Var}_{\pi^\circ}\left[p^\circ(\,\cdot\,,z)\right]}\right)\right)$$

We need to get estimates on mixing times and variances.
$\to$ use coalescent coupling and confined/tilted walk estimates.

Finally: determine the right time n, *i.e.* the number of excursions.

A sketch of the proof

First task: determine the invariant measures and show that $\pi^Y \approx \pi^Z$.
$\to$ use the ergodic theorem for the confined walk, and the independence structure of interlacements.

The error of the coupling is then at most

$$\sum_{\circ=Y,Z} \sum_{z\in\Sigma} \left(e^{-cn\varepsilon_N^2} + e^{-cn\varepsilon_N \frac{\pi^\circ(z)}{\nu^\circ(z)}} + \exp\left(-\frac{cn}{T^\circ_{\mathrm{mix}}} \frac{\pi^\circ(z)^2 \varepsilon_N^2 / k(\varepsilon_N)}{\mathrm{Var}_{\pi^\circ}\left[p^\circ(\,\cdot\,,z)\right]} \right) \right)$$

We need to get estimates on mixing times and variances.
$\to$ use coalescent coupling and confined/tilted walk estimates.

Finally: determine the right time n, *i.e.* the number of excursions.
$\to$ choose $u_N = u(t_N)$ to have $\mathcal{N}^{\mathrm{RW}}(t_N) \sim \mathcal{N}^{\mathrm{RI}}(u_N)$.

An application: covering time of the confined walk

An application: covering time of the confined walk

Covering time: $\mathfrak{C}_N(\Lambda_N) = \inf\{t \geq 1 \,:\, \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t)\}$.

An application: covering time of the confined walk

Covering time: $\mathfrak{C}_N(\Lambda_N) = \inf\{t \geq 1 \,:\, \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t)\}$.

Theorem (B. 2025+ – in preparation)

Recall $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d$. *In probability, as* $N \to +\infty$,

$$\mathfrak{C}_N(\Lambda_N) \sim c_d \alpha_\Lambda^{-1} N^d \log |\Lambda_N| \quad \textit{with} \quad \alpha_\Lambda = \inf_{x \in \Lambda} \varphi^2(x)\,,$$

φ *is the principal Laplace eigenfunction with Dirichlet boundary condition.*

An application: covering time of the confined walk

Covering time: $\mathfrak{C}_N(\Lambda_N) = \inf\{t \geq 1 \,:\, \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t)\}$.

Theorem (B. 2025+ – in preparation)

Recall $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d$*. In probability, as* $N \to +\infty$,

$$\mathfrak{C}_N(\Lambda_N) \sim c_d \alpha_\Lambda^{-1} N^d \log|\Lambda_N| \quad \textit{with} \quad \alpha_\Lambda = \inf_{x \in \Lambda} \varphi^2(x)\,,$$

φ *is the principal Laplace eigenfunction with Dirichlet boundary condition.*

The main reason: we can prove $|\phi_N - \varphi(\cdot/N)|_\infty \to 0$.

An application: covering time of the confined walk

Covering time: $\mathfrak{C}_N(\Lambda_N) = \inf\{t \geq 1 \, : \, \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t)\}$.

Theorem (B. 2025+ – in preparation)

Recall $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d$. *In probability, as* $N \to +\infty$,

$$\mathfrak{C}_N(\Lambda_N) \sim c_d \alpha_\Lambda^{-1} N^d \log |\Lambda_N| \quad \textit{with} \quad \alpha_\Lambda = \inf_{x \in \Lambda} \varphi^2(x)\,,$$

φ *is the principal Laplace eigenfunction with Dirichlet boundary condition.*

The main reason: we can prove $|\phi_N - \varphi(\cdot/N)|_\infty \to 0$.

The main idea: use coupling to study cover level of interlacements.

An application: covering time of the confined walk

Covering time: $\mathfrak{C}_N(\Lambda_N) = \inf\{t \geq 1 \,:\, \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t)\}$.

Theorem (B. 2025+ – in preparation)

Recall $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d$. *In probability, as* $N \to +\infty$,

$$\mathfrak{C}_N(\Lambda_N) \sim c_d \alpha_\Lambda^{-1} N^d \log |\Lambda_N| \quad \textit{with} \quad \alpha_\Lambda = \inf_{x \in \Lambda} \varphi^2(x)\,,$$

φ *is the principal Laplace eigenfunction with Dirichlet boundary condition.*

The main reason: we can prove $|\phi_N - \varphi(\cdot/N)|_\infty \to 0$.

The main idea: use coupling to study cover level of interlacements.
$\rightsquigarrow$ explicit law of covering time + decorrelation inequalities.

An application: covering time of the confined walk

Covering time: $\mathfrak{C}_N(\Lambda_N) = \inf\{t \geq 1 \,:\, \Lambda_N \subseteq \mathcal{R}_{\phi_N}(t)\}$.

Theorem (B. 2025+ – in preparation)

Recall $\Lambda_N = (N \cdot \Lambda) \cap \mathbb{Z}^d$. *In probability, as* $N \to +\infty$,

$$\mathfrak{C}_N(\Lambda_N) \sim c_d \alpha_\Lambda^{-1} N^d \log|\Lambda_N| \quad \textit{with} \quad \alpha_\Lambda = \inf_{x\in\Lambda} \varphi^2(x)\,,$$

φ *is the principal Laplace eigenfunction with Dirichlet boundary condition.*

The main reason: we can prove $|\phi_N - \varphi(\cdot/N)|_\infty \to 0$.

The main idea: use coupling to study cover level of interlacements.
$\rightsquigarrow$ explicit law of covering time + decorrelation inequalities.

Under stronger assumptions on Λ, we also get the convergence of the late points:

$$\sum_{x\in\Lambda_N} \delta_x \cdot \mathbb{1}\{H_x > c_d \alpha_\Lambda^{-1} N^d (\log|\Lambda_N| + s)\} \xrightarrow[N\to+\infty]{\mathcal{L}} \mathrm{PPP}(e^{-s}\mu_\lambda)\,.$$

About the eigenvector ϕ_N

About the eigenvector ϕ_N

Two motivations: understand the confined walk + replace ϕ_N with $\varphi(\cdot/N)$.

About the eigenvector ϕ_N

Two motivations: understand the confined walk + replace ϕ_N with $\varphi(\cdot/N)$. With Q. Berger, we investigated the regularity of ϕ_N and its rate of convergence towards φ.

About the eigenvector ϕ_N

Two motivations: understand the confined walk + replace ϕ_N with $\varphi(\cdot/N)$.
With Q. Berger, we investigated the regularity of ϕ_N and its rate of convergence towards φ.

Write $D_i f(x) = \frac{1}{2}\big(f(x+e_i) - f(x-e_i)\big)$ and $D_{i_1,\dots,i_k} = D_{i_1}\cdots D_{i_k}$.

About the eigenvector ϕ_N

Two motivations: understand the confined walk + replace ϕ_N with $\varphi(\cdot/N)$. With Q. Berger, we investigated the regularity of ϕ_N and its rate of convergence towards φ.

Write $D_i f(x) = \frac{1}{2}\big(f(x+e_i) - f(x-e_i)\big)$ and $D_{i_1,\dots,i_k} = D_{i_1}\cdots D_{i_k}$.

Theorem (Berger & B. 2025+ – arXiv:2408.15858)

There are constants $C_D, C'_D > 0$ such that, for any $k \geq 1$ and any $i_1,\dots,i_k \in \{1,\dots,d\}$, for any $x \in \Omega_N$ we have

$$|\phi_N(x)| \leq C_D \frac{d(x,\partial D_N)}{N} \quad , \quad \big|D_{i_1,\dots,i_k}\phi_N(x)\big| \leq \frac{(C'_D k)^k}{N^k}\left(\frac{d(x,\partial D_N)}{N}\right)^{1-k} .$$

About the eigenvector ϕ_N

Two motivations: understand the confined walk + replace ϕ_N with $\varphi(\cdot/N)$.
With Q. Berger, we investigated the regularity of ϕ_N and its rate of convergence towards φ.

Write $D_i f(x) = \frac{1}{2}\big(f(x+e_i) - f(x-e_i)\big)$ and $D_{i_1,\dots,i_k} = D_{i_1}\cdots D_{i_k}$.

Theorem (Berger & B. 2025+ – arXiv:2408.15858)

There are constants $C_D, C'_D > 0$ such that, for any $k \geq 1$ and any $i_1,\dots,i_k \in \{1,\dots,d\}$, for any $x \in \Omega_N$ we have

$$|\phi_N(x)| \leq C_D \frac{d(x,\partial D_N)}{N} \quad , \quad \big|D_{i_1,\dots,i_k}\phi_N(x)\big| \leq \frac{(C'_D k)^k}{N^k}\left(\frac{d(x,\partial D_N)}{N}\right)^{1-k} .$$

Theorem also works for $\varphi(\cdot/N)$, both versions use a Feynman-Kac representation of ϕ_N, φ.

About the eigenvector ϕ_N

Two motivations: understand the confined walk + replace ϕ_N with $\varphi(\cdot/N)$. With Q. Berger, we investigated the regularity of ϕ_N and its rate of convergence towards φ.

Write $D_i f(x) = \frac{1}{2}\big(f(x+e_i) - f(x-e_i)\big)$ and $D_{i_1,\dots,i_k} = D_{i_1}\cdots D_{i_k}$.

Theorem (Berger & B. 2025+ – arXiv:2408.15858)

There are constants $C_D, C'_D > 0$ such that, for any $k \geq 1$ and any $i_1,\dots,i_k \in \{1,\dots,d\}$, for any $x \in \Omega_N$ we have

$$|\phi_N(x)| \leq C_D \frac{d(x,\partial D_N)}{N} \quad , \quad \big|D_{i_1,\dots,i_k}\phi_N(x)\big| \leq \frac{(C'_D k)^k}{N^k}\left(\frac{d(x,\partial D_N)}{N}\right)^{1-k} .$$

Theorem also works for $\varphi(\cdot/N)$, both versions use a Feynman-Kac representation of ϕ_N, φ.

Main tool: "multi-mirror" coupling in order to pair 2^k SRW/BM so that each pair coalesce before exiting the ball B_R with probability $\geq 1 - R^{-k}$.

Thank you for your attention!