Localization of a polymer in a random environment - Bernoulli & Interlacements

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joint work with Quentin Berger & Julien Poisat

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$k_B := 1.380649 \times 10^{-23}\,\mathrm{kg.m^2/s^2.K}$ is the Boltzmann constant.

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Polymer measure: Gibbs transformation of $\mathbf{P}$ given by

$$\mathbf{P}_n^\beta(S) = \frac{1}{Z_n^\beta} e^{-\beta H_n(S)} \mathbf{P}(S)$$

with $\beta = 1/k_B T$ and $Z_n^\beta := \mathbf{E}\left[e^{-\beta H_n(S)}\right]$ the partition function.

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$$\mathbb{P}_n^{\beta}(S) = \frac{1}{\mathbb{Z}_n^{\beta}} \mathbb{E}\left[e^{-\beta H_n^\omega(S)}\right] \mathbf{P}(S)$$

where the environment plays a part in the equilibrium (most probable and most favorable realizations).

LPSM

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$$\mathbb{P}_n^p(S) = \frac{1}{\mathbb{Z}_n^p} \mathbb{E}_p \left[\mathbb{1}_{\{S \cap \mathcal{O}_p = \varnothing\}} \right] \mathbf{P}(S) = \frac{1}{\mathbb{Z}_n^p} e^{|\mathcal{R}_n(S)| \log(1-p)} \mathbf{P}(S)$$

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$$\mathbb{P}^{h,\beta}_{n,\omega}(S) = \frac{1}{\mathbb{Z}^{h,\beta}_{n,\omega}} \prod_{z\in\mathcal{R}_n(S)} (1-p_z)\mathbf{P}(S) = \frac{1}{\mathbb{Z}^{h,\beta}_{n,\omega}} e^{\sum_{z\in\mathcal{R}_n(S)}(\beta\omega_z - h)}\mathbf{P}(S)$$

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This model is the random walk penalized by the sum of $h - \beta\omega_z$ sitting on the range, with quenched ω_z.

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$$\mathbf{P}_{n,h}^{\omega,\beta}(S) := \frac{1}{Z_{n,h}^{\omega,\beta}} \exp\Big(\sum_{z\in\mathcal{R}_n(S)} (\beta\omega_z - h)\Big)\mathbf{P}(S)\,.$$

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We are interested in the edges localization, meaning asymptotics for

$$M_n^- = \min_{k\leq n} S_k \;,\; M_n^+ = \max_{k\leq n} S_k$$

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- *Assume that* $n^{-1/2}(\log n)^{3/2} \leq h_n \ll n^{1/4}$*, then*

$$\left(\frac{|\mathcal{R}_n| - R_n^*}{a_n}, \frac{M_n^+}{R_n^*}\right) \xrightarrow[n\to+\infty]{(d)} (\mathcal{R}, \mathcal{M}),$$

where $\mathcal{R} \sim \mathcal{N}(0,1)$ *and* $\mathcal{M} \sim \frac{\pi}{2}\sin(\pi y)\mathbb{1}_{[0,1]}(y)dy$ *are independent.*

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- *Assume that* $n^{1/4} \ll h_n \ll n$*, then.*

$$\forall n \geq 1, \exists \mathcal{A}_n \subseteq \{0,1\}, \quad \lim_{n\to\infty} \mathbf{P}_{n,h_n}\left(|\mathcal{R}_n| - \lfloor R_n^* - 2 \rfloor \notin \mathcal{A}_n\right) = 0.$$

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Theorem (Berger, Huang, Torri, Wei (2022))

For all $h > 0$, define $c_h := (\pi^2 h^{-1})^{1/3}$. Then, for any $h, \beta > 0$, $\mathbb{P}$-almost surely we have the following convergence

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \log Z_{n,h}^{\omega,\beta} = -\frac{3}{2}(\pi h)^{2/3}, \qquad n^{-1/3}|\mathcal{R}_n| \xrightarrow[n\to\infty]{\mathbf{P}_{n,h}^{\omega,\beta}} c_h\,.$$

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LPSM

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The main contribution to $Z_{n,h}^{\omega,\beta}$ is given by trajectories with range $\sim c_h n^{1/3}$.

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For any $h, \beta > 0$, we have the following $\mathbb{P}$-a.s. convergence

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Furthermore, $u_* := \arg\max_{u\in[0,c_h]} \{X_u^{(1)} + X_{c_h-u}^{(2)}\}$ is $\mathbb{P}$-a.s. unique and

$$n^{-1/3}(M_n^-, M_n^+) \xrightarrow[n\to\infty]{\mathbf{P}_{n,h}^{\omega,\beta}} (-u_*, c_h - u_*) \qquad \mathbb{P} - a.s.$$

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The location of the range is **P**-deterministic: it only depends on the realization ω through $X_u := X_u^{(1)} + X_{c_h-u}^{(2)}$.

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Theorem

Suppose $\mathbb{E}\left[|\omega_0|^{3+\delta}\right] < \infty$ *for some positive* δ*, we have the* $\mathbb{P}$*-a.s. convergence*

$$\lim_{n\to\infty} \frac{\sqrt{2}}{\beta n^{1/9}} \left(\log Z_{n,h}^{\omega,\beta} + \frac{3}{2} h c_h n^{1/3} - \beta n^{1/6} X_{u_*} \right) = \sup_{u,v} \left\{ \mathcal{Y}_{u,v} - c_{h,\beta} (u+v)^2 \right\}$$

where $\mathcal{Y}_{u,v} := \mathbf{Y}_u - \mathbf{Y}_{-v} - \chi(\mathbf{B}_u + \mathbf{B}_v)$ *with* $\mathbf{B}$ *a two-sided* BES_3*,* $\mathbf{Y}$ *a two-sided standard Brownian motion.*

LPSM

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What is the scale/law of $U_n := M_n^- + u_* n^{1/3}$ or $V_n := (c_h - u_*)n^{1/3} - M_n^+$?

Theorem

Suppose $\mathbb{E}\left[|\omega_0|^{3+\delta}\right] < \infty$ *for some positive* δ*, we have the* $\mathbb{P}$*-a.s. convergence*

$$\lim_{n\to\infty} \frac{\sqrt{2}}{\beta n^{1/9}} \left(\log Z_{n,h}^{\omega,\beta} + \frac{3}{2} h c_h n^{1/3} - \beta n^{1/6} X_{u_*} \right) = \sup_{u,v} \left\{ \mathcal{Y}_{u,v} - c_{h,\beta} (u+v)^2 \right\}$$

where $\mathcal{Y}_{u,v} := \mathbf{Y}_u - \mathbf{Y}_{-v} - \chi(\mathbf{B}_u + \mathbf{B}_v)$ *with* $\mathbf{B}$ *a two-sided* BES_3*,* $\mathbf{Y}$ *a two-sided standard Brownian motion.*

Moreover, $(\mathcal{U}, \mathcal{V}) := \arg\max_{u,v}\{\mathcal{Y}_{u,v} - c_{h,\beta}(u+v)^2\}$ *is* $\mathbb{P}$*-a.s. well-defined, unique, and we have*

$$\frac{1}{n^{2/9}} (U_n, V_n) \xrightarrow[n\to\infty]{\mathbf{P}_{n,h}^{\omega,\beta}} (\mathcal{U}, \mathcal{V}) \quad \mathbb{P} - a.s.$$

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Again, this is **P**-deterministic.

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For this talk: focus on the annealed measure.

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$\rightarrow$ what about capacity ?

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for $d \geq 5$, $\gamma = \frac{1}{d-2}$ is critical $\to$ volume at dimension $d-2$!!!

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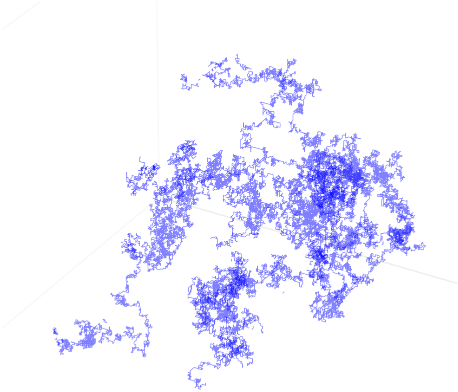

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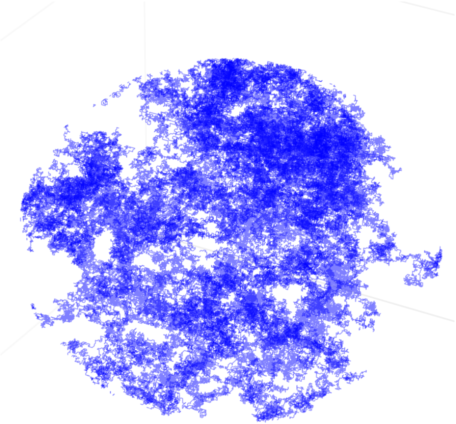

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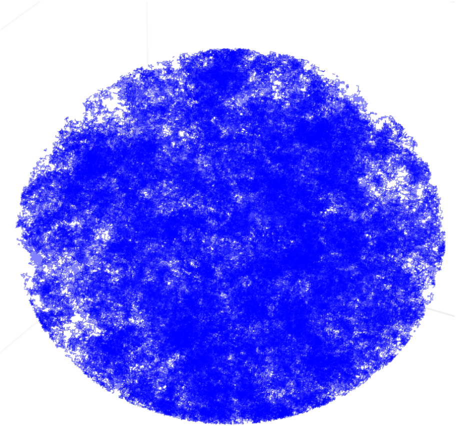

Thank you!